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ON BICOMPACT SEMIGROUPS

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We shall investigate in this note the structure of a minimal ideal of a bicomact semigroup, and to extend the theory of Suschkewitsch's kernel [1] (which he calls "Kerngruppe") of finite semigroups to bicomact semigroups.

If S is a bicomact semigroup, then S has a minimal ideal K , and K is completely simple in the sense of Rees [2] and is decomposed into groups which are isomorphic one another and have no element in common. Further, we shall show that minimal ideals (left, right and two-sided) of S are bicomact and closed in S . If S contains zero element, then K is zero alone, while if S has no zero, then K is a completely simple semigroup without zero.

1. A set S is called a *semigroup*, if in S a single-valued product ab is defined for every pair a, b of S such that for product the associative law holds:

$$(ab)c = a(bc).$$

By a *sub-semigroup* of S we mean a non-vacuous subset A of S with the property $A^2 \subset A$, i.e. $ab \in A$ for every a, b in A . By a *left ideal* of S we mean a non-vacuous subset L of S such that $SL \subset L$. Analogously, we can define a *right ideal* R of S . If M is both a left and a right ideal of S , then M is called a (*two-sided*) *ideal* of S .

An element e of S is called an *idempotent*, if $e^2 = e$. An element 0 is termed *zero*, if $0x = 0 = x0$ for all x in S . Then it will easily be seen that the zero of S , if it exists, is uniquely defined and is an idempotent. An element 1 is termed the *identity* of S , if $1x = x = x1$ for all x in S . Then the identity of S , if it exists, is uniquely defined, and is an idempotent.

2. If S is a semigroup and at the same time it is a topological space (in this note a topological space means a Hausdorff space), and moreover, the multiplicative operation in the semigroup S is continuous in the topological space S , then S is called a *topological semigroup*. If a sub-semigroup T of S is closed (open) in the space S , then we shall call T a *closed (open) sub-semigroup* etc.

It is clearly to be seen that a sub-semigroup T of S is itself a topological semigroup under the relative topology.

If a topological semigroup S is bicomact as a topological space, S is called a *bicomact semigroup*.

Let T_1, T_2 be two bicomact subsets of a topological semigroup S , then $T_1 T_2$ is also bicomact. For let us consider the product space $T_1 \times T_2$, then by Tychonoff's theorem $T_1 \times T_2$ is a bicomact topological space, and $T_1 T_2$ is a continuous image of $T_1 \times T_2$. From this it follows that $T_1 T_2$ is bicomact. In particular, if S is bicomact, and $a \in S$, then aS, Sa, S^2, \dots are all bicomact and closed in S .

3. Lemma 1. *Let S be a topological semigroup and B^* be a bicomact subset of S . Let $A = \{\lambda\}$ be an index system and $A = \{a_\lambda; a_\lambda \in S, \lambda \in A\}$, $B = \{b_\lambda; b_\lambda \in S, \lambda \in A\}$ be subsets of S whose elements correspond to the same index system A . Moreover, we suppose that $B \subset B^*$ and $a \in \bar{A}$. Then there exists $b \in \bar{B}$ such that $ab \in \bar{C}$, where $C = \{a_\lambda b_\lambda; \lambda \in A\}$.*

Proof. We denote by $\Sigma_a = \{V_\tau(a); \tau \in T\}$ a complete system of neighborhoods of the element a . And put $A_\tau = V_\tau(a) \cap A$, then since $a \in \bar{A}$, A_τ is not empty. By B_τ we denote the set of elements of B , whose elements have the same indices with those of elements of A_τ . Let $\mathfrak{B} = \{B_\tau; \tau \in T\}$, then \mathfrak{B} is a family of subsets of B^* with the finite intersection property. For, let $B_{\tau_1}, \dots, B_{\tau_n}$ be any finite number of sets in \mathfrak{B} , and $A_{\tau_i}, i = 1, \dots, n$, are the corresponding subsets of A . Then, since $V_{\tau_i}(a), i = 1, 2, \dots, n$, are neighborhoods of the element a and $\Sigma_a = \{V_\tau(a); \tau \in T\}$ is a complete system of neighborhoods of a , there exists a neighborhood $V_{\tau_0}(\in \Sigma_a, \tau_0 \in T)$ of a such that $V_{\tau_0}(a) \subset \bigcap_{i=1}^n V_{\tau_i}(a)$. Let $A_{\tau_0} = V_{\tau_0}(a) \cap A$ and denote by B_{τ_0} the subset in \mathfrak{B} which corresponds to A_{τ_0} , then it is clear that

$$\phi \neq B_{\tau_0} \subset \bigcap_{i=1}^n B_{\tau_i}.$$

Thus, \mathfrak{B} has the finite intersection property. And since B^* is bicomact we have $\bigcap_{\tau \in T} \bar{B}_\tau \neq \phi$. Let $b \in \bigcap_{\tau \in T} \bar{B}_\tau$ and $V(ab)$ any neighborhood of ab , then there exist neighborhoods $V_\kappa(a)$ of a and $V(b)$ of b such that $V_\kappa(a) V(b) \subset V(ab)$, where $V_\kappa(a) \in \Sigma_a, \kappa \in T$. Since $b \in \bigcap_{\tau \in T} \bar{B}_\tau, b \in \bar{B}_\kappa$ and $V(b) \cap B_\kappa \neq \phi$. Let b_{κ_0} be any element of $V(b) \cap B_\kappa$, then, since

$b_{\kappa_1} \in B_{\kappa}$, there exists an element a_{κ_0} such that $a_{\kappa_0} \in A_{\kappa_0} = V_{\kappa}(a) \cap A$. Hence $a_{\kappa_0} b_{\kappa_0} \in V_{\kappa}(a) V(b) \subset V(ab)$. On the other hand $a_{\kappa_0} b_{\kappa_0} \in C$. Hence $V(ab) \cap C \neq \phi$. Thus, $ab \in \bar{C}$.

In a semigroup S , if $ax = ay$ ($xa = ya$) implies $x = y$ for every a, x, y in S , then S is called a semigroup *satisfying the left (right) cancellation law*. S is called a semigroup *satisfying the cancellation law*, if it satisfies both the left and the right cancellation law.

Lemma 2L. *Let S be a bicomact semigroup satisfying the left cancellation law, and B be a closed subset of S . If $p \in S$, $pB \subset B$, then $pB = B$.*

Proof. From the assumption, we have $B \supset pB \supset p^2B \supset \dots$. Put $P_{\nu} = \{p^i; i \geq \nu\}$ and $\mathfrak{P} = \{P_{\nu}; \nu = 1, 2, \dots\}$, then it is clear that \mathfrak{P} is a family of subsets of a bicomact space S with the finite intersection property. Hence

$$\bigcap_{\nu=1}^{\infty} \bar{P}_{\nu} \neq \phi.$$

Let q be any element of $\bigcap_{\nu=1}^{\infty} \bar{P}_{\nu}$. Then we shall show that $\bigcap_{i=1}^{\infty} p^i B = qB$. Let qx ($x \in B$) any element of qB and $V(qx)$ be an arbitrary neighborhood of qx , then there exists a neighborhood $V(q)$ of q such that $V(q)x \subset V(qx)$. Since $q \in \bigcap_{\nu=1}^{\infty} \bar{P}_{\nu}$, $V(q) \cap P_{\nu} \neq \phi$ for $\nu = 1, 2, \dots$. Therefore, an integer i_{ν} exists so that $p^{i_{\nu}+1} \in V(q)$ for $\nu = 1, 2, \dots$. Hence, $p^{i_{\nu}+1}x \in V(q)x \subset V(qx)$. On the other hand, $p^{i_{\nu}+1}x \in p^{i_{\nu}+1}B \subset p^{\nu}B$. Hence $V(qx) \cap p^{\nu}B \neq \phi$ for $\nu = 1, 2, \dots$, and so $qx \in \bar{p^{\nu}B} = p^{\nu}B$ for $\nu = 1, 2, \dots$. This shows that $qB \subset \bigcap_{i=1}^{\infty} p^i B$.

Conversely, if p' be any element of $\bigcap_{i=1}^{\infty} p^i B$, then p' can be written in the form $p' = p^i b_i$, $i = 1, 2, \dots$, where $b_i \in B$. Now, let $B' = \{b_i; i = 1, 2, \dots\}$ and $P = \{p^n; n = 1, 2, \dots\}$, then by Lemma 1 there exists an element $b \in B' \subset B$ such that $qb \in \overline{\{p^i b_i; i = 1, 2, \dots\}} = \bar{p'} = p'$. Thus $p' = qb \in qB$. This shows that $\bigcap_{i=1}^{\infty} p^i B \subset qB$. And we have $\bigcap_{i=1}^{\infty} p^i B = qB$.

Analogously, if we replace B by pB , we can conclude $\bigcap_{i=1}^{\infty} p^i (pB) = q(pB)$. Since $\bigcap_{i=1}^{\infty} p^i B = \bigcap_{i=1}^{\infty} p^i (pB)$, we have $qB = q(pB)$. Applying the left cancellation law, it follows that $pB = B$. This proves Lemma 2L.

Similarly, we obtain the following two lemmas:

Lemma 2R. *Let S be a bicomact semigroup satisfying the right cancellation law, and B be a closed subset of S . If $p \in S$, $Bp \subset B$, then $Bp = B$.*

Lemma 2. *Let S be a bicomact semigroup satisfying the cancellation law, and B be a closed subset of S . If $p \in S$, $Bp \subset B$ and $pB \subset B$, then $Bp = B = pB$.*

From Lemma 2 it follows

Theorem 1. *A bicomact semigroup satisfying the cancellation law is a group.*

4. Lemma 3. *Let S be a bicomact semigroup and a an element of S and let $A = \{a^n; n = 1, 2, \dots\}$. Then \bar{A} contains a commutative closed group D .*

Proof. Let $A_\nu = \{a^i; i \geq \nu\}$ and $\mathfrak{A} = \{A_\nu; \nu = 1, 2, \dots\}$. Then in the same way with Lemma 2L $D = \bigcap_{\nu=1}^{\infty} \bar{A}_\nu \neq \phi$.

Now we shall show that D is a commutative closed group. It is easy to see that D is a commutative closed sub-semigroup of S . It remains to show that D forms a group. To prove this it is sufficient to show that $xD = D$ for all x in D . Suppose that there exists y in D such that $yD \subsetneq D$, then there is z in D so that $z \notin yD$, that is, $z \neq yx_\lambda$ for every x_λ in D . Therefore, there exist neighborhoods $V_\lambda(y)$ of y , $V(x_\lambda)$ of x_λ and $V_\lambda(z)$ of z such that $V_\lambda(z) \cap V_\lambda(y)V(x_\lambda) = \phi$. Since $\bigcup_{x_\lambda \in D} V(x_\lambda) \supset D$ and D is bicomact as a closed subset of a bicomact space S , we can choose a finite covering $V(x_{\lambda_i})$ ($i = 1, 2, \dots, k$) of D , i.e. $\bigcup_{i=1}^k V(x_{\lambda_i}) \supset D$. Let $V(y)$, $V(z)$ be neighborhoods of y , z , respectively, such that $V(y) \subset \bigcap_{i=1}^k V_{\lambda_i}(y)$, $V(z) \subset \bigcap_{i=1}^k V_{\lambda_i}(z)$, and put $\bigcup_{i=1}^k V(x_{\lambda_i}) = Q$, then Q is an open set containing D , and

$$V(z) \cap (V(y)Q) = \phi.$$

Since $y \in D$, there is an integer $\mu \geq 1$ so that $a^\mu \in V(y)$, and since $z \in D$, there exist integers ν_i such that $\nu_i > \mu$, $\nu_{i+1} > \nu_i$ ($i = 1, 2, \dots$) and $a^{\nu_i} \in V(z)$ for every ν_i . Putting $\tau_i = \nu_i - \mu \geq 1$ and $\bar{A}^{(\tau_i)} = \{a^j; j = t, t+1, \dots\}$, then, by the above method, $\bigcap_{i=1}^{\infty} \bar{A}^{(\tau_i)} \neq \phi$. And it is easily shown that $\bigcap_{i=1}^{\infty} \bar{A}^{(\tau_i)} \subset D$. Choose an element u from $\bigcap_{i=1}^{\infty} \bar{A}^{(\tau_i)}$,

then since Q is an open set containing D , there exists a neighborhood $V(u)$ of u contained in Q , and since $u \in \bigcap_{t=1}^{\infty} \overline{A^{(\tau_t)}}$, there exists an integer τ_k so that $a^{\tau_k} \in V(u)$. Then, $a^{\nu_k} \in V(z)$ and $a^{\nu_k} = a^{\mu+\tau_k} = a^{\mu} a^{\tau_k} \in V(y) V(u) \subset V(y)Q$. This contradicts to $V(z) \cap V(y)Q = \phi$. Hence, we obtain $xD = D$ for all x in D , and D is a group.

As a consequence of Lemma 3 we have

Lemma 4. *Every bicomact semigroup has at least one idempotent.*

5. If a semigroup S contains no proper ideal at all, it is called a *simple* semigroup. An idempotent f is said to be *under* another one e if $ef = f = fe$. An idempotent e is *primitive* if there is no non-zero idempotent under e . A simple semigroup S is said to be *completely simple* if every idempotent element of S is primitive, and for each $a \in S$ there exist idempotents e and f such that $ea = a = af$.

Lemma 5. *A necessary and sufficient condition for semigroup S to be simple is that $SxS = S$ for all x of S .*

Lemma 6. *If S is a simple semigroup and e is an idempotent of S , then eSe is also a simple semigroup.*

The above two lemmas can be proved in the same way with [2].

Lemma 7. *A bicomact simple semigroup S is completely simple.*

Proof. Let a be any element of S . Then, since S is simple, there exist b, c such that $bac = a$. Then, by simple induction, $b^na c^n = a$ for all integers n . Let $B = \{b^n; n = 1, 2, \dots\}$, $B_{\nu} = \{b^i; i \geq \nu\}$ and $D_b = \bigcap_{\nu=1}^{\infty} \overline{B_{\nu}}$, then by Lemma 3, D_b is a commutative closed subgroup of S . Analogously, let $C = \{c^n; n = 1, 2, \dots\}$, $C_{\nu} = \{c^i; i \geq \nu\}$ and $D_c = \bigcap_{\nu=1}^{\infty} \overline{C_{\nu}}$, then D_c is also a commutative closed sub-group of S . We denote by e_b and e_c the identities of the groups D_b and D_c , respectively. Then, we shall show first that $e_b a c' = a$, where $c' \in \overline{C}$.

Let $H = \{b^na; n = 1, 2, \dots\}$. In Lemma 1, if we replace A by H , B by C , A by $\{1, 2, \dots\}$, B^* by S and a by $e_b a$, then it is easy to see that all the conditions of Lemma 1 are satisfied for H , C and $e_b a$. Hence from Lemma 1 we can conclude that there exists $c' \in \overline{C}$ such that $e_b a c' \in \{\overline{b^na c^n}; n = 1, 2, \dots\} = \{\overline{a}\} = a$. This shows that a can be written in the form $a = e_b a c'$, where $c' \in \overline{C}$. Analogously, a can be written in the form $a = b' a e_c$, where $b' \in \overline{B}$. Hence

$$\begin{aligned} e_h a &= e_h(e_h a c') = (e_h e_h)(a c') = e_h a c' = a \\ a e_e &= (b' a e_e) e_e = (b' a)(e_e e_e) = b' a e_e = a \end{aligned}$$

and e_h, e_e are idempotent.

Let e be any idempotent of S , and f be an idempotent under e . Then $f = efe \in eSe$, and by Lemma 6 eSe is simple, so that there exist x, x' in eSe such that $xfx' = e$. Putting $xf = y$ and $fx' = y'$, we obtain $yfy' = e$, $yf = y$ and $fy' = y'$. Then $yy' = yfy' = e$. Then, by induction, since $yfy' = e$, we have $y^n f y'^n = e$ for all integers n . Then as above proved, we can choose idempotents g, h in eSe such that $gfh' = e$ and $g'fh = e$, where g' and h' are contained in $\{y^n; n = 1, 2, \dots\}$ and $\{y'^n; n = 1, 2, \dots\}$, respectively. Then, since e is the identity of eSe , we have

$$\begin{aligned} g &= ge = g \cdot gfh' = gfh' = e, \\ fh' &= e \cdot fh' = gfh' = e, \end{aligned}$$

henceforth

$$f = f \cdot e = f \cdot fh' = fh' = e.$$

Thus, e is the only idempotent contained in eSe , and e must be primitive. (In the latter half of the proof of this lemma, we limited ourselves in a bicomact semigroup eSe). Hence S is completely simple.

Theorem 2. *A bicomact semigroup S has the unique minimal two-sided ideal K which is completely simple and bicomact.*

Proof. Let E be the set of all idempotents in S , then by Lemma 4, E is not empty. We denote by e_λ, e_μ, \dots the elements of E . Then, it is clear that the set $K = \bigcap_{e_\lambda \in E} Se_\lambda S$ is a closed bicomact ideal of S , if $K \neq \phi$. Now, let $Se_{\lambda_i} S$ ($i = 1, 2, \dots, m$) be any finite number of subsets in the family $\{Se_\lambda S; e_\lambda \in E\}$. Then each $Se_{\lambda_j} S$ ($1 \leq j \leq m$) contains the element

$$e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_m} = (e_{\lambda_1} \dots e_{\lambda_{j-1}}) e_{\lambda_j} (e_{\lambda_{j+1}} \dots e_{\lambda_m})$$

so that $\bigcap_{i=1}^m Se_{\lambda_i} S \neq \phi$. Thus, $\{Se_\lambda S; e_\lambda \in E\}$ is a family of subsets of a bicomact space S with the finite intersection property. Since each $Se_\lambda S$ is closed, it holds

$$K = \bigcap_{e_\lambda \in E} Se_\lambda S \neq \phi.$$

Suppose that K' is a bicomact ideal of S . Then, by Lemma 4, K' contains an idempotent e' and it follows easily that

$$K \subset Se'S \subset K'.$$

If M is a minimal ideal of S then SaS is a bicomact ideal of S contained in M where a is an element M . SaS being bicomact, it follows

$$K \subset SaS \subset M,$$

Therefore, $K = M$ because M is a minimal ideal of S .

Now, let x be any element of K . Then, KxK is obviously a bicomact ideal of S contained in K . On the other hand, KxK must contain K . Hence, $KxK = K$. This shows that K is simple and consequently, it may contain no other ideal of S than itself. Therefore K is the minimal ideal of S .

In the following, the minimal ideal K which is completely simple and bicomact is called the "kernel (Suschkewitsch's Kerngruppe)" of S .

Since the kernel K of S is a bicomact semigroup it contains an idempotent. One can easily see that for any idempotent e of K the relation

$$SeS = K.$$

Epecially, if S contains 0 then, by definition, K contains also 0. Since 0 is an idempotent, it follows immediately

$$K = S0S = 0.$$

6. Lemma 8. *A completely simple semigroup S with the identity 1 is a group.*

Proof. Let x be an element of S . Then, since S is simple, there exist elements b, c such that $bxc = 1$. Then, xcb, cbx are idempotents. But, by the definition of a completely simple semigroup, 1 is a primitive idempotent and so must be the only idempotent of S . Hence $xcb = cbx = 1$, and every element x has the inverse $x^{-1} = cb$. Therefore, S is a group.

Lemma 9. *Let K be the kernel of a bicomact semigroup S . Then, for any idempotent e of K , eK and Ke satisfy the left and right*

cancellation law respectively. Moreover, for arbitrary idempotents e, f , eKf is a group.

Proof. Let x_1, x_2, y be elements of eK , and assume that

$$yx_1 = yx_2.$$

Then, we can determine the elements k_1, k_2, k of K such that $x_1 = ek_1, x_2 = ek_2$ and $y = ek$. Since eKe is a completely simple semigroup with e as the identity, then eKe is a group. Therefore, the element eke has the inverse element $(eke)^{-1}$ in eKe . From the relation $yx_1 = yx_2$, it follows immediately that

$$x_1 = ek_1 = (eke)^{-1}(eke k_1) = (eke)^{-1}(eke k_2) = ek_2 = x_2.$$

This shows that the left cancellation law holds in eK . Similarly, we may prove that in Ke the right cancellation law holds.

Now, it is not hard to show that eKf satisfies the left and right cancellation law. Since eKf is a bicomact semigroup, it must be a group by Theorem 1.

Theorem 3. *Let K be the kernel of a bicomact semigroup S , then K is decomposed into join of groups which have no element in common.*

Proof. Let E' be the set of all idempotents in K , and we denote by $G_{\lambda\mu}$ a group $e_\lambda Ke_\mu$, where e_λ, e_μ belong to E' . Since K is a completely simple semigroup, then by the definition of a completely simple semigroup, every a of K is contained in one of the groups $G_{\lambda\mu}$. Hence, it follows that

$$K = \bigcup_{\lambda, \mu} G_{\lambda\mu}.$$

If two groups $G_{\lambda\mu}$ and $G_{\kappa\tau}$ have an element c in common, then

$$e_1 c = c = e_2 c,$$

where e_1, e_2 are the identities of $G_{\lambda\mu}, G_{\kappa\tau}$ respectively. By multiplication with the inverse c_1^{-1} of c in the group $G_{\lambda\mu}$ from the right side, we have

$$e_1 = e_2 e_1,$$

Analogously,

$$e_2 = e_2 e_1.$$

Henceforth, $e_1 = e_2 = e$.

Then, $G_{\lambda\mu} = eG_{\lambda\mu}e \subset eKe = e_\lambda e_\mu (eKe) e_\lambda e_\mu \subset e_\lambda Ke_\mu = G_{\lambda\mu} (e_\lambda e_\mu \in eKe \text{ and } eKe \text{ is a group})$, hence $G_{\lambda\mu} = eKe$. Similarly, $G_{\kappa\tau} = eKe$. Thus, $G_{\lambda\mu} = eKe = G_{\kappa\tau}$. Hence, either $G_{\lambda\mu} = G_{\kappa\tau}$ or $G_{\lambda\mu} \cap G_{\kappa\tau} = \phi$.

Theorem 4L. *Let K be the kernel of a bicomact semigroup S , then*

(1) *$L = Ke$ is a minimal left ideal of S , where e is an idempotent in K .*

(2) *every minimal left ideal L of S can be expressed in the form $L = Ke$, where e is an idempotent in K .*

Proof. (1) Since L is bicomact and satisfies the right cancellation law by Lemma 9, then we obtain from Lemma 2R $Lp = L$ for every p in L .

Now, L' be a left ideal of S contained in L , then $L = Lp \subset L'$ for $p \in L'$, and then $L = L'$. Thus, L is minimal.

(2) Let L be a minimal left ideal of S . Then for every element a of L , Ka is a left ideal of S contained in L so that $L = Ka$. Henceforth, L is bicomact, and by Lemma 4, L has an idempotent e . Thus, $L = Ke$.

Analogously, we have

Theorem 4R. *Let K be the kernel of a bicomact semigroup S , then*

(1) *$R = eK$ is a minimal right ideal of S , where e is an idempotent in K .*

(2) *every minimal right ideal R of S can be expressed in the form $R = eK$, where e is an idempotent in K .*

Corollary. *A bicomact semigroup has at least one left and one right minimal ideals.*

Corollary. *Every minimal ideal (left, right and two-sided) of a bicomact semigroup is closed and bicomact.*

Theorem 5. *Let R and L be a right and a left minimal ideals of a bicomact semigroup S , respectively, and K be the kernel of S . Then $LR = K$ and RL is a group.*

This theorem is clear, by Theorems 4R and 4L.

Lemma 10L. *If e and f are two idempotents in K , then either $Ke = Kf$ or $Ke \cap Kf = \phi$.*

By Theorem 4L, Ke , Kf are minimal left ideals of S . Hence, it is easy to see that $Ke = Kf$ or $Ke \cap Kf = \phi$.

Similarly, we have

Lemma 10R. *If e and f are two idempotents in K then either $eK = fK$ or $eK \cap fK = \phi$.*

Theorem 6. *The kernel K of a bicomact semigroup S is the set theoretical join of all minimal left (or right) ideals of S .*

Proof. By Theorem 4L, K contains all minimal left (right) ideals of S . Now, let a be an element of K . Then by Theorem 3, there exist idempotents e_λ, e_μ of K such that $e_\mu Ke_\lambda$ contains a . Since $e_\mu Ke_\lambda \subset Ke_\lambda$, then by Theorem 4L, Ke_λ is a minimal left ideal of S containing a . Thus, K is the set theoretical join of all minimal left ideals of S .

Similarly, one can prove that K is the set theoretical join of all minimal right ideals of S .

Theorem 7. *The groups $G_{\lambda\mu} = e_\lambda Ke_\mu$ are isomorphic one another.*

Proof. By Lemmas 10L, 10R and Theorem 6, each group $G_{\lambda\mu}$ is contained in one and only one minimal left ideal Ke_μ and right ideal $e_\lambda K$. Then the idempotents in Ke_μ are right identities of $G_{\lambda\mu}$ and the idempotents in $e_\lambda K$ are left identities of $G_{\lambda\mu}$, and isomorphisms of the groups $G_{\lambda\mu}$ can be established as the same way with that of Suschkewitsch.

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